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Improving the signal–to–noise–ratio in lattice gauge theories

Kurt Langfeld, Hugo Reinhardt and Oliver Tennert

Institut für Theoretische Physik, Universität Tübingen
D–72076 Tübingen, Germany.

Abstract

Renormalization of composite fields is employed to suppress the statistical noise in lattice gauge calculations. We propose a new action which differs from the standard Wilson action by "irrelevant" operators, but suppresses the fluctuations of the plaquette. We numerically study the Creutz ratios and find a scaling window. The SU(2) mass gap is estimated. We prove that the contributions of the "irrelevant" operators to the screening mass decrease towards the continuum limit. The results obtained from the action with noise suppression are compared with those of the standard Wilson action.

renormalization of composite fields, lattice gauge theory, signal–to–noise–ratio

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1 Introduction

The present lattice calculations provide the only rigorous approach to low energy Yang-Mills theories. After the scaling window had been discovered by Creutz in his pioneering work [1], lattice simulations provided first informations on the ratio of the low lying glue-ball masses and the string tension for the SU(2) [2] and the SU(3) [3, 4] gauge group.
Unfortunately, the limited capacity of the computers put severe constraints on the accuracy of "lattice measurements". Firstly, the finite number of lattice links correspond to a finite physical volume. Nowadays, a physical volume of $(1.6\,\mathrm{fm})^4$ is available for reasonable values of the lattice spacing (see e.g. [4]). Secondly, the finite number of independent configurations which are employed to calculate the expectation value of the desired operator implies that the "lattice measurement" is contaminated with statistical noise.
Weisz and Symanzik have shown that the situation corresponding to the finite size problem can be significantly improved by using an improved action [5]. In the numerical simulation, an effective action is used which already contains corrections from perturbative radiation. In recent years, much work has been devoted to the development of such improved lattice actions, which are often referred to as "perfect" lattice actions [6].
In this paper, we will focus on the noise problem. In order to outline the conceptual nature of the noise problem, we briefly review the arguments presented in [7]. Glueball (screening) masses m_g are extracted from correlation functions, i.e.

$$C(t) := \langle \Phi(t)\, \Phi(0) \rangle \;\approx\; const.\; e^{-m_g t} \,, \tag{1}$$

where the brackets indicate an average over independent lattice configurations. The statistical error of the desired quantity is measured by the standard deviation [7]

$$\langle \Phi(t)\Phi(0)\Phi(t)\Phi(0) \rangle \;-\; C^2(t) \;\approx\; \langle \Phi^2(0) \rangle \,. \tag{2}$$

From perturbation theory, one knows that composite operators acquire new divergences implying that the statistical error of the correlation function $C(t)$, i.e. $\sqrt{\langle \Phi^2(0) \rangle}$, diverges in the continuum limit $a \to 0$. The disastrous and fundamental problem therefore is that the signal–to–noise–ratio vanishes in the continuum limit.
In the recent past, two concepts have been established to be important in order to improve the signal–to–noise–ratio. Firstly, the choice of a non-local operator $\Phi(x)$ in (1) (which nevertheless carries the quantum numbers of the state under investigation) might result in a composite operator $\Phi^2(x)$ which is free of ultraviolet divergences (*smearing* [8]). Secondly, informations from the links of a former update step is used to enhance the signal (*fuzzing* [9]). Hybrid algorithms, which combine "smearing" and "fuzzing", as well as an estimate of their impact on the signal–to–noise–ratio can be found in [7].

In this paper, we prose a new method to improve the signal–to–noise–ratio. The method is inspired from the renormalization procedure for composite operators in continuum quantum field theory. The basic idea is to add to the Wilson action an additional term which vanishes faster than the Wilson action in the continuum limit, i.e. we add an "irrelevant operator", but which suppresses the statistical noise. We study the efficiency of our method by calculating the SU(2) mass gap employing the "old" idea of plaquette–plaquette correlations.
The paper is organized as follows: in the next section, we briefly review the renormalization of composite operators in continuum quantum field theory. We then discuss the modifications of the Wilson action by "irrelevant" composite fields which yield the suppression of the noise. In the third section, we present our numerical results. Conclusions are left to the final section.

2 Renormalization of composite operators

2.1 In continuum quantum field theory

For illustration purposes, we here consider the continuum quantum field theory of a field $\phi(x)$ which is described by the Euclidean partition function

$$Z[\eta](g) \;=\; \int \mathcal{D}\phi(x)\; \exp\left\{ -S[\phi](g) + \int d^4x\, \eta(x)\phi(x) \right\} \;, \tag{3}$$

where a regularization is understood in order to make (3) well defined. For simplicity, we assume that the Euclidean action S contains only one parameter g (e.g. the coupling strength). The external source η linearly couples to the field $\phi(x)$ implying that functional derivatives of $Z[\eta](g)$ with respect to η yield connected Green's functions

$$\langle T\, \phi(x_1) \ldots \phi(x_N) \rangle \;. \tag{4}$$

These Green's functions are generically divergent in four space-time dimensions, if the regulator is removed. Renormalized Green's functions are obtained from the generating functional

$$Z_R[\eta_R](g_R) \;:=\; Z[Z_\eta \eta_R](Z_g g_R) \tag{5}$$

by performing the functional derivative with respect to η_R. The Zs are the so-called renormalization constants, and η_R is the renormalized source which accounts for field renormalization, and g_R is the renormalized parameter. In order to guarantee that (5) yields finite Green's functions, we have tacitly assumed that the field theory (3) is multiplicative renormalizable [10], i.e. all divergences can be absorbed in the renormalization constants Z.
The crucial observation is that, if we allow for composite field insertions, i.e. if we are interested in the limit $x_1 \to x_2$ in the renormalized Green's function, new divergences

arise in $Z_R[\eta_R](g_R)$. In order to renormalize these insertions, we generalize

$$Z_R[\eta_R](g_R) \to Z_R[\eta_R, j_R](g_R) \; = \tag{6}$$
$$\int \mathcal{D}\phi(x) \; \exp\left\{-S[\phi](Z_r g) + \int d^4x \; \left[Z_\eta \eta_R(x)\phi(x) + Z_j j_R(x)\phi^2(x)\right]\right\} \; .$$

The additional divergences due to the composite field $\phi^2(x)$ can be absorbed in the renormalization constant Z_j.

The dependence of the renormalization constants on the regulator is of course not known a priori. Perturbation theory usually provides a systematic way to extract this dependence [10]. In the context of numerical lattice gauge calculations, the question arises, how one should choose the bare source $j(x)$ as function of the lattice spacing in order to renormalize the composite field insertions and therefore to reduce the statistical noise in the continuum limit. We will answer this question in the next section.

2.2 In lattice gauge calculations

The partition function of SU(2) lattice Yang-Mills theory is defined as a functional integral over the link variables $U_\mu(x)$, i.e.

$$Z \; = \; \int \mathcal{D}U_\mu \; \exp\{-S\} \; , \tag{7}$$

where the standard Wilson action is given by

$$S_W \; = \; \sum_{\{x\}\mu\nu} \beta \; [1 \; - \; P_{\mu\nu}(x)] \; . \tag{8}$$

$P_{\mu\nu}$ is the plaquette, which is built from four link variables, i.e.

$$P_{\mu\nu}(x) \; = \; \frac{1}{2}\mathrm{tr}\left\{U_\mu(x)U_\nu(x+\mu)U^\dagger_\mu(x+\nu)U^\dagger_\nu(x)\right\} \; . \tag{9}$$

The functional integral (7) is defined on a lattice with lattice spacing a, which serves as the ultraviolet regulator. In the continuum limit ($a \to 0$), the plaquette is

$$P_{\mu\nu}(x) \; = \; 1 \; - \; \frac{a^4}{4} F^a_{\mu\nu} F^a_{\mu\nu} \; + \; \mathcal{O}(a^6) \; , \tag{10}$$

where $F^a_{\mu\nu}$ is the usual field strength tensor. In this paper, we will confine ourselves to the plaquette-plaquette correlation function $\langle P_{\mu\nu}(x) P_{\alpha\beta}(0)\rangle$ in order to extract the mass gap of the SU(2) lattice theory. From (2), it is clear that this correlation function is plagued by a statistical noise which diverges in the continuum limit. From the discussions in the last section, it is now evident that one must add a term

$\sum_{\{x\}\mu\nu} j(x) P^2_{\mu\nu}(x)$ with a suitable choice of $j(x)$ to the action (8) in order to avoid this divergence. We here propose to perform the numerical simulation using the action

$$S \;=\; \sum_{\{x\}\mu\nu} \beta \, [1 \,-\, P_{\mu\nu}(x)] \;+\; \sum_{\{x\}\mu\nu} j \; [P_{\mu\nu}(x) - \mathcal{A}]^2 \;, \tag{11}$$

where j and $\mathcal{A}$ are constants. In fact, we will choose $\mathcal{A}$ to be the average value of the plaquette $P_{\mu\nu}$, which will be a function of β and j.
Let us study the naive continuum limit of the action S (11). Using (10), a direct calculation yields (up to a constant)

$$S \;=\; [\beta \,-\, 2j\,(1-\mathcal{A})] \; \frac{a^4}{4} F^a_{\mu\nu} F^a_{\mu\nu} \;+\; \mathcal{O}(a^6) \; . \tag{12}$$

This implies that the action S cannot be distinguished in the naive continuum limit from Wilson's action (8) with an effective parameter $\beta_{eff} \,=\, \beta - 2j(1-\mathcal{A})$. In the quantum continuum limit ($\beta \to \infty$), the average plaquette and effective inverse temperature β_{eff} are approximately given by [1]

$$\mathcal{A} \;=\; 1 - \frac{3}{4\beta} \; , \qquad \beta_{eff} \;=\; \beta - \frac{3j}{2\beta} \; . \tag{13}$$

If j increases less than linearly with β, the results of the quantum theory using S should agree with those which are obtained by employing the Wilson action.
On the other hand, the term in (11) proportional to j further constrains the plaquette to its average value $\mathcal{A}$ and therefore suppresses statistical fluctuations around the average value of the plaquette. The key point is that the action S (11) differs from the Wilson action by "irrelevant" operators which are chosen to suppress the statistical noise of the plaquette. The prize one has to pay is that the average plaquette value must already be known at the beginning of the numerical simulation.

3 Numerical results

3.1 Creutz ratios

We perform our numerical simulations of the quantum theory employing the action (11) on a 10^4 lattice using the heat bath algorithm by Creutz [1]. Our purpose is to demonstrate the mechanism of noise suppression proposed in the previous sections, rather than to provide new precision measurements. In the latter case, one should resort to "improved" actions [5] as well as a larger number of lattice points.
The first task is to calculate the average plaquette $\mathcal{A}$ self-consistently for given values for β and j. We apply the following procedure (we leave it to the reader to develop his own method): before the lattice has reached its thermo-dynamical equilibrium,

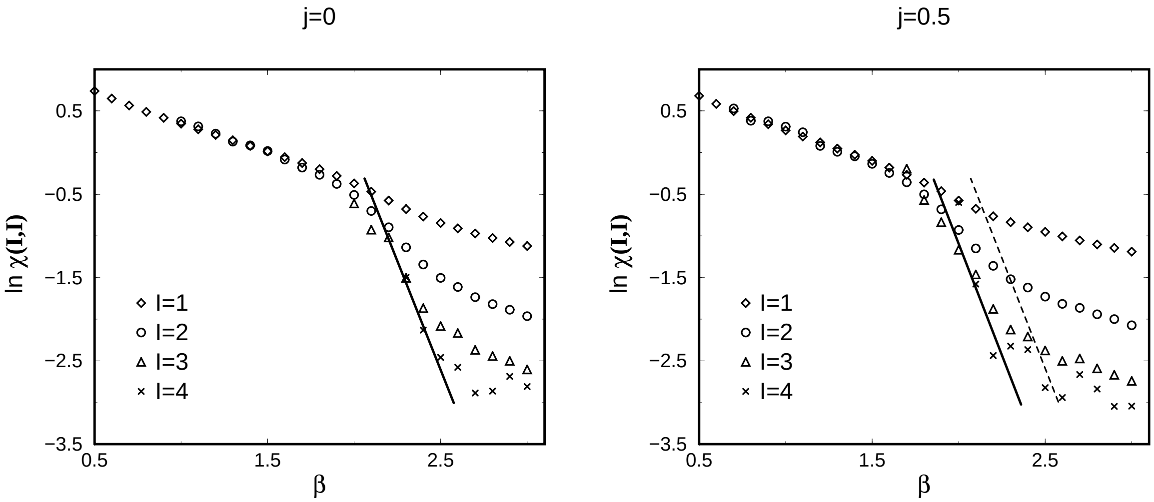

Figure 1: The Creutz ratios employing the standard Wilson action (left) and using the action S (11) with $j = 0.5$ (right). The dashed line in the right hand picture indicates the perturbative scaling behavior of the case $j = 0$.

we use the lattice average from the previous heat bath step for $\mathcal{A}$. When equilibrium is reached, we calculate the average plaquette taking into account all heat bath steps. In a particular step, we assign the actual value of this average plaquette to $\mathcal{A}$. We then perform a large number of heat bath steps to obtain an accurate value of $\mathcal{A}$, which subsequently enters the numerical calculations of correlation functions, where a smaller number of heat bath steps is sufficient.

The crucial question is whether the scaling limit is reached for finite values of j. In order to answer this question, we calculate the Creutz ratios [1] as a function of β. The left picture of figure 1 shows the case $j = 0$. These are the Creutz ratios which one obtains using the standard Wilson action. These results are compared with those for $j = 0.5$ (in the right picture of figure 1). The lines indicate the scaling behavior which has been calculated with the help of the perturbative renormalization group β-function. The crucial observation is that the model with non-vanishing source j also approaches the scaling behavior (shown by the lines in figure 1) which is predicted by the perturbative renormalization group. Also note that the perturbative scaling already sets in at smaller values of β compared with those of the case $j = 0$. This is precisely what one expects from (13).

3.2 Noise suppression

Let us study the efficiency with which the action in (11) suppresses the statistical error of the average value of the plaquette. For this purpose we consider the probability distribution of finding a particular value of the plaquette $P_{\mu\nu}$ in the interval

$$[\mathcal{A} - 5 \times 10^{-3},\ \mathcal{A} + 5 \times 10^{-3}] \,. \tag{14}$$

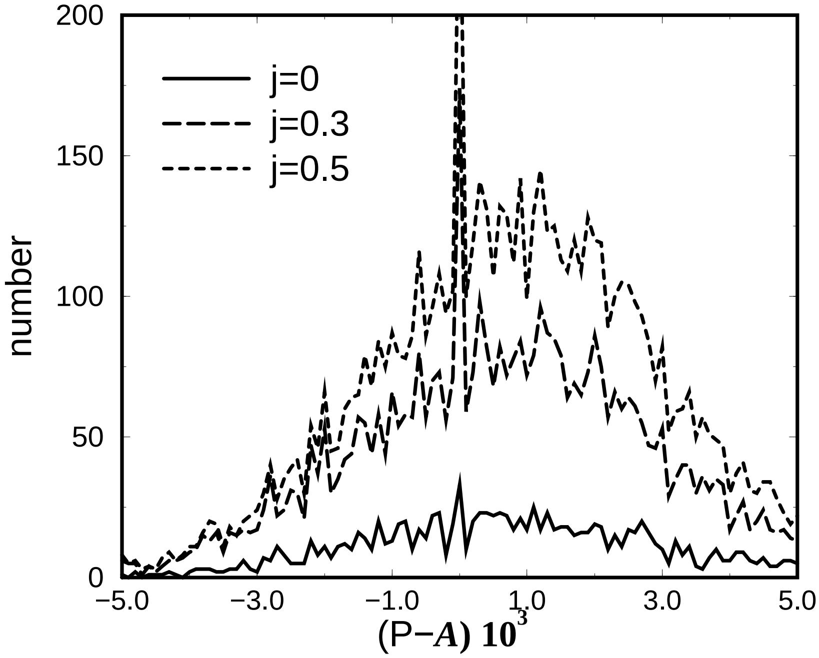

Figure 2: The distribution of the plaquette around its average value $\mathcal{A}$ for several values of the noise suppression factor j.

From the numerical point of view, we proceed as follows: we divide the above interval in bins of width 10^{-4}, and calculate the lattice average of the plaquette in a particular heat bath step. We then count the number of average values which correspond to a certain bin. We evaluated 1140 heat bath steps. The numerical result is shown in figure 2 for $\beta = 2.1$ and $j = 0.5$. One clearly observes that the data points are strongly grouped around the corresponding average value $\mathcal{A}$ for large values of the noise suppression factor j. In addition, one observes a sharp peak at $P_{\mu\nu} = \mathcal{A}$. Whether this peak is an artifact due to corrections to the action of order a^8 or whether the peak is necessary to reproduce the standard Yang-Mills action (12) in the scaling limit $\beta \to \infty$, is not clear to us.

3.3 The SU(2) mass gap

In this subsection, we will numerically estimate the SU(2) mass gap from the plaquette-plaquette correlation function in order to demonstrate how the action (11) works in practice. The purpose of this subsection is two-fold: Firstly, we want to show that the mass gap obtained here is in agreement with the high precision measurements [2] which employ high statistics and an improvement of the signal–to–noise–ratio using the fuzzing technique. Secondly, we will compare the results at

several values of β in order to estimate the contribution of the "irrelevant" operators off the continuum limit. We are aware of the problem that the overlap of the plaquette with glue-ball wave-function is small [11]. For high precision measurements, one should therefore employ non-local operators. Performing the noise suppression for the case of these operators, however, might be numerically costly. For these first investigations, we therefore confine ourselves to the study of correlations of the plaquette.

Furthermore, we use the source method [12] to calculate the correlation function of the plaquettes. For this purpose, we estimate the average plaquette at the origin from

$$W[\eta] \;=\; \frac{\int \mathcal{D}U_\mu \;\sum_{\mu\nu} P_{\mu\nu}(0)\;\exp\left\{-S + \sum_{\{x\}\mu\nu} \eta_x P_{\mu\nu}(x)\right\}}{\int \mathcal{D}U_\mu \;\;\exp\left\{-S + \sum_{\{x\}\mu\nu} \eta_x P_{\mu\nu}(x)\right\}} \tag{15}$$

where S is the action (11). Let $P(x)$ denote $\sum_{\mu\nu} P_{\mu\nu}(x)$. It is straightforward to verify that the functional derivative of $W[\eta]$ with respect to the source $\eta(x)$ yields the desired correlation function, i.e.

$$C(t) \;=\; \frac{\delta W[\eta]}{\delta\eta(x)}|_{\eta=0} \;=\; \langle P(x)P(0)\rangle \;-\; \langle P(x)\rangle\langle P(0)\rangle \;. \tag{16}$$

In practice, we are interested in the correlation of the plaquette in time direction implying that one chooses $\eta(x) = \eta(t)$. In fact, one simulates two statistical ensembles. One ensemble is generated with the inverse temperature, the other is obtained by setting the inverse temperature of the time slice $t = 0$ to $\beta+\eta$ leaving the remaining β-values unchanged [12]. In both ensembles, the average plaquette is obtained as a function of time. The correlation function (16) is obtained by approximating the functional derivative in (16) by the difference of the expectation values of the plaquette of the two ensembles.

We use 1140 heat bath steps to extract the average plaquette in both ensembles. A typical result for the correlation function as a function of time is shown in figure 3, where the noise suppression factor is set to $j = 0.5$. The inverse temperature is $\beta = 2.1$ guaranteeing that the systems are in the scaling region (see left picture of figure 1). One clearly observes an exponential decay of the correlation, where the slope of $\ln C(t)$ provides the SU(2) mass gap in units of the lattice spacing.

It is interesting to compare the value of the mass gap, obtained with and without noise suppression, for β-values corresponding to the middle and the onset of the scaling window, respectively. In the latter case, the contribution of "irrelevant operators" to the mass gap should be more pronounced compared with the former case. The numerical results are summarized in the following table.

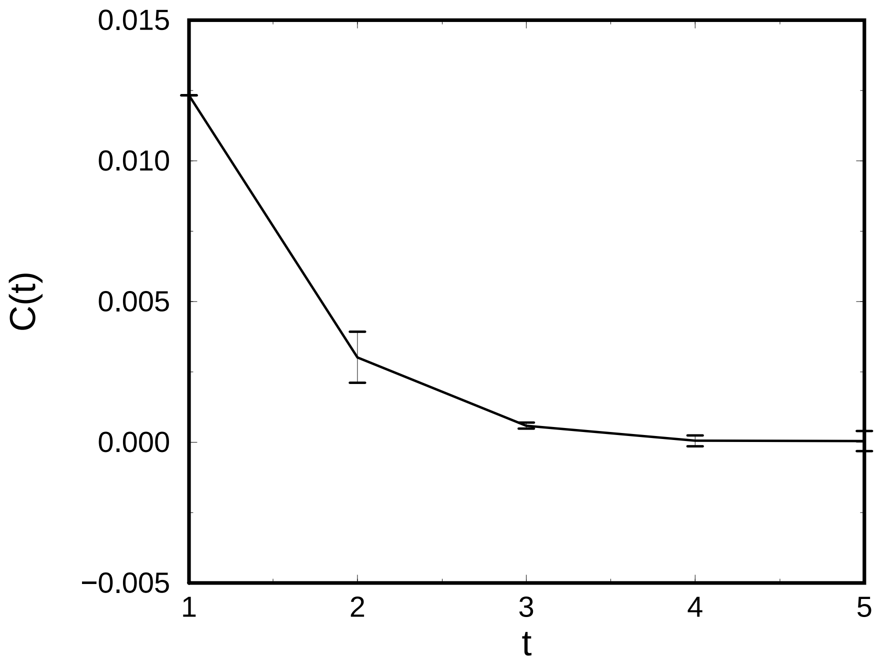

Figure 3: The correlation function (16) as a function of time t in units of lattice spacings for $\beta = 1.9$ and $j = 0.5$.

β	j	κa^2	L	$m\,a$	$m/\sqrt{\kappa}$
2.1	0	0.577	3.5 fm	1.40 ± 0.14	1.85 ± 0.2
1.9	0.5	0.577	3.5 fm	1.56 ± 0.23	2.06 ± 0.3
2.3	0	0.21	2.1 fm	1.36 ± 0.03	2.97 ± 0.06
2.1	0.5	0.21	2.1 fm	1.46 ± 0.06	3.25 ± 0.13
2.4	0	0.126	1.6 fm	1.43 ± 0.08	4.04 ± 0.3
2.2	0.5	0.115	1.5 fm	1.41 ± 0.07	4.14 ± 0.2

The error bars indicate the uncertainty due to statistical fluctuations. They are extracted from the fit of the correlation function $C(t)$ to the function $const.\ \exp\{-mt\}$. The string tension κ sets the scale. We use $\kappa = 440\,$MeV. L is the extension of our lattice in each direction. One observes that the discrepancy between the values $m/\sqrt{\kappa}$ with and without noise suppression decreases, if the ensemble turns towards the continuum limit, i.e. κa^2 decreases. This shows that the influence of the "irrelevant" operators, which distinguishes our action (16) from the standard Wilson action diminishes. Note, however, that at physical lattice sizes where the influence of the "irrelevant" operators is small, finite size effects might play a role. This would imply that a large number of lattice points (here we use 10^4 lattice points) is necessary for high precision measurements of the SU(2) mass gap. The numerical result for the mass gap in the scaling region is in agreement with the results of [2].

4 Conclusions

We have shown that constraining the plaquette to its average value can be understood as composite field renormalization. The parameter j, which enters the action, acts as a renormalization constant absorbing the divergences arising from the composite nature of fields. These divergences are responsible for the small signal–to–noise–ratio in correlation functions of plaquettes. The action (11) with noise suppression differs from the standard Wilson action up to a shift in β only by "irrelevant operators", i.e. both actions coincide in the naive continuum limit.
We have numerically studied the new action (11) which implements the noise suppression on a coarse grained lattice consisting of 10^4 lattice points. We have used a β independent noise suppression factor j. Other choices are possible and perhaps more convenient depending on the type of correlation function which is under consideration.
We have extracted the Creutz ratios from the numerical data with and without noise suppression and have established a scaling window in both cases. We have numerically confirmed that the action (11) reproduces the scaling of the standard Wilson action with β shifted to a lower value in agreement with the analytical result. The SU(2) mass gap has been estimated from the plaquette–plaquette correlation function. We have found that the contributions from the "irrelevant" operators to the screening mass decrease with increasing values of β. The goal of the noise suppression has mainly been the reduction of the statistical fluctuations of the "background", on top of which the signal exists.
For high precision measurements of glue-ball masses, one should use correlation functions of operators which have a larger overlap with the glue-ball wave function than the plaquettes. In addition, "perfect" actions will help to extrapolate to the continuum limit. A generalization of the noise suppression introduced in the present paper to either case seems feasible. In the case of the "perfect" actions, one has to ensure that the noise suppression term does not spoil the correct ultra-violet behavior exploited by the "perfect" action technique.

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